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Polarization Calculations for Electron Storage Rings*

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Abstract

A computer program called SMILE has been developed to calculate the equilibrium polarization in a high-energy electron storage ring. It can calculate spin resonances to arbitrary orders, in principle. Results of polarization calculations are shown for a variety of storage ring models, to elucidate various aspects of the behaviour of the polarization, such as the effects of machine symmetry, beam energy spread, and transverse momentum recoils, etc. Reasonable agreement is obtained with some experimental data from measurements at SPEAR.

I. INTRODUCTION

The chief goal of polarization calculations is to calculate the equilibrium polarization, including spin resonances of arbitrary order (at least in principle). A formula for the equilibrium polarization in an electron storage ring was given in 1973 by Derbenev and Kondratenko.[1] (Some theoretical aspects were clarified in Ref. 2.) However, no practical algorithm was given in Ref. 1 for evaluating the polarization formula for a given storage ring model. Such an algorithm has been developed in Refs. 3 and 4, to calculate spin resonances to arbitrary orders (in principle) in the approximation of treating only linear orbital dynamics, and a computer program called SMILE[4] has been written to implement it. Results of polarization calculations for a variety of models are presented below, to elucidate various aspects of the behaviour of the polarization in a high-energy electron storage ring. Reasonable agreement is obtained with some experimental data from measurements at SPEAR.

Various topics are beyond the scope of this report. In particular, SMILE is a program to calculate the polarization for a given storage ring model. It is not, in its present form, a fitting program. Thus, it does not perform closed orbit corrections or spin matching. These topics will not be discussed below. SMILE also assumes the beam is in equilibrium. It does not calculate the time evolution of the orbital and/or spin distribution, e.g. by tracking of a set of electrons. Such calculations are also beyond the scope of this report.

II. GENERAL REMARKS

The equilibrium degree of radiative polarization in a high-energy electron storage ring is given by the Derbenev-Kondratenko formula[1]

$$P_{eq} = \frac{8}{5\sqrt{3}} \frac{\left\langle |\rho|^{-3}\, \hat{b}.\left[\hat{n} - \gamma\frac{\partial \hat{n}}{\partial \gamma}\right]\right\rangle}{\left\langle |\rho|^{-3}\left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 + \frac{11}{18}\left|\gamma\frac{\partial \hat{n}}{\partial \gamma}\right|^2\right]\right\rangle} . \tag{1}$$

Here $\vec{v}$ is the particle velocity, γ is the particle energy in units of rest mass energy, $\hat{b} \equiv \vec{v} \times \dot{\vec{v}}/|\vec{v} \times \dot{\vec{v}}|$, $\hat{n}$ is the spin quantization axis and ρ is the local radius of curvature of the particle trajectory. The angular brackets denote an equilibrium ensemble average over

the distribution of particle orbits and the ring azimuth. The corresponding polarization build-up time, τ_{pol}, is given by

$$\tau_{pol}^{-1} = \left\langle \frac{5\sqrt{3}}{8} \frac{e^2 \hbar \gamma^5}{m^2 c^2} \frac{1}{|\rho|^3} \left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 + \frac{11}{18} \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 \right] \right\rangle , \tag{2}$$

where m and e are the particle mass and charge, respectively. The axis $\hat{n}$ depends on the particle orbit, $\hat{n} = \hat{n}(\vec{r}, \vec{p}, \theta)$, and, by definition, satisfies the Thomas-BMT equation

$$\frac{d\hat{n}}{d\theta} = \vec{\Omega}(\vec{r}, \vec{p}, \theta) \times \hat{n} \tag{3}$$

and also the periodicity conditions[1,2]

$$\hat{n}(I, \psi, \theta) = \hat{n}(I, \psi + 2\pi, \theta) = \hat{n}(I, \psi, \theta + 2\pi) , \tag{4}$$

where $\{I, \psi\}$ denote the orbital action-angle variables, and $\vec{\Omega}$ is the spin precession vector. Because there is a distribution of vectors $\hat{n}$, the full equilibrium polarization vector $\vec{P}_{eq}$ is given by (see Ref. 3 for details)

$$\vec{P}_{eq} = \langle \vec{s}.\hat{n} \rangle \langle \hat{n} \rangle . \tag{5}$$

The Derbenev-Kondratenko formula gives the value of $\langle \vec{s}.\hat{n} \rangle$ only:

$$P_{DK} = \langle \vec{s}.\hat{n} \rangle . \tag{6}$$

The contribution of the $\langle \hat{n} \rangle$ term will be investigated below. It turns out to be negligible.

A computer program called SMILE has been written to calculate $\hat{n}$ and $\gamma(\partial\hat{n}/\partial\gamma)$ in the approximation of linear orbital dynamics, and results are presented below. Details of the computer algorithm are given in Ref. 4, but some features of the program will be described briefly here. The program uses a recursive algorithm, and is therefore able to calculate the polarization up to arbitrary orders of spin resonances without requiring new code to be written for each new order. The formalism uses perturbation theory, and the perturbation expansion parameter is the orbital amplitude (strictly, there are three expansion parameters, because there are three amplitudes, but they are all treated on the same footing). The user can instruct the program to calculate to different orders in the various modes, thus one can, for example, calculate to higher-order in the synchrotron oscillations

than in the betatron oscillations, if desired. The program can also accept storage ring lattices of arbitrary geometry, including magnet misalignments and spin rotators (or other non-planar lattices). It uses a fully symplectic 6-dimensional formalism to calculate the orbital motion. Overlapping spin resonances can be calculated without requiring any special treatment. The beam energy spread, as well as the transverse beam sizes, are taken into account when calculating the equilibrium polarization. In the results presented below, it is assumed that the orbital distribution is Gaussian in all planes. If desired, a more general orbital distribution can be implemented into the code.

II. RESULTS

A. PERFECTLY ALIGNED MACHINE

Various results of polarization calculations using SMILE[4] will now be given. Graphs of the the polarization as a function of $a\gamma$, where $a = (g - 2)/2$, will be shown. In units of energy, $a\gamma = E(GeV)/.440652$. In all the graphs that appear below, the symbol ν denotes the spin tune, while the orbital tunes are Q_x, Q_z and Q_s, in standard notation. In this section we consider results from a perfectly aligned storage ring. In Fig. 1, the polarization is shown for a two-fold symmetric machine, with vertical bends in the lattice. The calculation is to fourth order in all orbital modes. The spin resonances $\nu = 2 + Q_s$, $\nu = 2 + Q_z$, and $\nu = 4 - Q_z$, with their satellites, are visible. Note that, because of the machine symmetry, odd harmonics of the spin resonances do not appear. In Fig. 2, the same graph is shown in more detail in the vicinity of the resonance $\nu = 2 + Q_z$. It can be seen that the second order resonances $\nu = 2 + Q_z - Q_s$ and $\nu = 4 - 2Q_z$ overlap. No special modification is required to calculate the polarization in this region — it is not necessary, in the SMILE algorithm, to assume that the resonances are widely separated.

B. BEAM ENERGY SPREAD

The ensemble averages in Eq. (1) require that the orbital beam emittances, e.g. the beam energy spread, be taken into account when calculating the polarization. To show that this is done in the SMILE program, the polarization was recalculated for the same storage ring model used above, but all the emittances were multiplied by a factor of 10

(so the beam energy spread was increased by $\sqrt{10}$). The result is shown in Fig. 3, in the same spin-tune range as Fig. 2, and we see that the resonances are indeed stronger. In particular, the overlapping resonances $\nu = 2 + Q_z - Q_s$ and $\nu = 4 - 2Q_z$ can no longer be resolved — they appear as one resonance.

C. THE FACTOR $\langle \hat{n} \rangle$ IN THE POLARIZATION

It was stated above that there is a contribution to the polarization, viz., the factor $\langle \hat{n} \rangle$ in Eq. (5), that is not given by the Derbenev-Kondratenko formula (Eq. (1)). In Fig. 4, the contribution of this factor to the polarization is calculated for the same model used above. First, we write

$$\hat{n} = \hat{n}_0 \sqrt{1 - |\zeta|^2} + \text{Re}(\zeta \vec{k}_0) \qquad (7)$$

where $\hat{n}_0$ is the value of $\hat{n}$ on the closed orbit, and $\vec{k}_0$ is a vector orthogonal to $\hat{n}_0$. Its detailed definition does not matter here. Then, to first order in the orbital motion, we can write

$$\langle \hat{n} \rangle = \hat{n}_0 \langle \sqrt{1 - |\zeta|^2} \rangle + \text{Re}(\langle \zeta \rangle \vec{k}_0)$$
$$\simeq \hat{n}_0 (1 - \frac{\langle |\zeta|^2 \rangle}{2}) + (\text{higher order}) . \qquad (8)$$

In Fig. 4, the value of $\log_{10} \langle |\zeta|^2 \rangle$ is shown along with the value of P_{DK} from Fig. 2, for only the lowest order contribution to the polarization. We see that, although in principle the factor $\langle \hat{n} \rangle$ can lead to large depolarization not given by the Derbenev-Kondratenko formula, in practice

$$\langle |\zeta|^2 \rangle < 10^{-7} \qquad (9)$$

throughout almost the whole graph, and even near the center of the resonance $\langle |\zeta|^2 \rangle < 10^{-2.5}$. Barber[5] has reported similar results for HERA, showing that this factor is unimportant in storage rings of very different energy and circumference.

D. VERTICAL FLUCTUATIONS

The derivation of the Derbenev-Kondratenko formula contains the approximation that the electron recoil due to a photon emission is only longitudinal (i.e. only the effects of

energy loss are treated). Transverse recoil terms are neglected. In a perfectly aligned horizontal storage ring, it turns out that energy loss terms do not excite any spin resonances, whereas the vertical recoils do, so that there are corrections to the Derbenev-Kondratenko formula in a perfectly aligned horizontal storage ring. For a weak-focusing ring, it is reported[6] that the polarization can increase to 99.2% near a spin resonance: the vertical recoil terms do not necessarily always decrease the polarization. The formula giving the combined effect of recoils along the electron momentum (longitudinal) and in the direction of the field (transverse) is[7]

$$
P_{eq} = \frac{8}{5\sqrt{3}} \frac{\left\langle \frac{1}{|\rho|^3} \left[\hat{n}.\hat{b} - \gamma \frac{\partial \hat{n}}{\partial \gamma}.\hat{b} + \frac{1}{3\gamma} \frac{\partial \hat{n}}{\partial \beta_b}.\hat{v} \right] \right\rangle}{\left\langle \frac{1}{|\rho|^3} \left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 + \frac{11}{18} \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 + \frac{1}{9\gamma} \frac{\vec{v}}{|\vec{v}|}.\left(\hat{n} \times \frac{\partial \hat{n}}{\partial \beta_b} \right) + \frac{13}{90\gamma^2} \left| \frac{\partial \hat{n}}{\partial \beta_b} \right|^2 \right] \right\rangle} . \quad (10)
$$

Here $\gamma(\partial \hat{n}/\partial \gamma)$ describes the effects energy loss, due to a photon emission, and $\partial \hat{n}/\partial \beta_b$ describes the effects of recoil in the direction of the local magnetic field. Algorithms to calculate these vectors are given in Refs. 4 and 7. Eq. (10) has been evaluated for various perfectly aligned horizontal strong-focusing storage rings, and results are given in Ref. 8. A sample graph from Ref. 8 is shown in Fig. 5. The calculation is again to leading order. The resonance is $\nu = Q_z$, and the polarization increases to approximately 98% on one side of the resonance. However, this result is quite sensitive to the presence of closed orbit distortions in the ring, as is shown in the next section.

E. CLOSED ORBIT DISTORTIONS

In Fig. 6, the polarization is calulated to second order in all modes, for the same model used in Fig. 5. The r.m.s. vertical closed orbit distortion is 1.1 mm. The solid curve shows the result of using only Eq. (1), and the dashed curve shows the result of using Eq. (9). There is difference of about 2% in magnitude between the curves, near the resonance $\nu = Q_z$, and the difference decreases away from this resonance.

The relative importance of transverse recoils decreases with increasing energy (see Ref. 7). This is illustrated in Fig. 7, where the polarization is calculated with (dashed curve) and without (solid curve) vertical momentum recoils for a horizontal storage ring

with a 1 mm r.m.s. vertical closed orbit distortion. The two curves are indistinguishable on the scale of this graph.

The polarization was also calculated for the model used in Figs. 1 and 2, with an r.m.s. vertical closed orbit distortion of 1 mm, and the result is shown in Fig. 8. The calculation was again to fourth order in all modes. This time we see resonances such as $\nu = 3 + Q_x$ and $\nu = 3 \pm Q_s$, which were forbidden by symmetry in the perfectly aligned lattice.

The SMILE program was also applied to two verions of the PEP storage ring, and the results are shown in Figs. 9 and 10. In both Figures, the dot-dash curve is the zeroth order calculation, the dotted curve is the second order result, and the solid curve is the fourth order result. (Because the orbital distribution is assumed to be a Gaussian symmetric about the closed orbit, the contributions of other orders vanish in the ensemble average. The same feature is displayed by all the other graphs in this report.) Fig. 9 was obtained using a mini-beta PEP lattice, and Fig. 10 was obtained by using an SSRL lattice. In both cases the r.m.s. vertical closed orbit distortion was 0.7 mm. No orbit correction or spin matching techniques were used. The first-order spin integrals are identified in the Figures. We see that the ring lattice (for a given set of magnets and ring geometry) has a profound influence on the polarization. In the SSRL lattice, the resonances are not only weaker, but there are also fewer resonances, than in the mini-beta lattice. Unfortunately, for high-energy physics experiments, a mini-beta lattice is required.

A graph of polarization vs. energy was obtained in experiments performed at SPEAR.[9] The graph is reproduced in Fig. 11. The curve is a guide to the eye, not a theoretical fit. The orbital tunes are labelled ν_x, ν_y and ν_s, instead of Q_x, Q_z and Q_s, respectively. Recently, a result has been reported[10] of a calculation of the ratio of the width of the second order resonance $\nu = 3 + Q_x - Q_s$ to that of the first order resonance $\nu = 3 + Q_x$ at 3.65 GeV in Fig. 11, which agrees with the experimental result. This new calculation has caused some doubt to be expressed[10] about the validity of previous calculations of higher-order resonances,[11] which are claimed not to agree with the experimental result.

A storage ring model with the approximate propeties of SPEAR was used to see if the SMILE algorithm could also explain the ratio of the widths of these resonances.

The parameters of the model were adjusted to fit the width of the first-order resonance $\nu = 3 + Q_x$, and the widths of the two satellites $\nu = 3 + Q_x - Q_s$ and $\nu = 3 + Q_x - 2Q_s$ were calculated without further adjustment of the model. The result is shown in Fig. 12. The solid curve is the SMILE result, and the dashed curve is the experimental curve from Fig. 11. We see that there is reasonable agreement between experiment and theory. Thus it appears that there is not necessarily any contradiction between the new calculation[10] and the SMILE formalism.

One should note also that the data shown Fig. 11 were not obtained in one experimental run, but over several runs, and the machine properties (such as the tunes) varied between runs.[12] Adjustments to compensate for these effects were made in Ref. 8, but one must therefore be careful in attempting a detailed comparison between theory and experiment (Fig. 11). For this reason, one should not necessarily conclude that Fig. 11 contradicts calculations such as Ref. 11. In particular, the SMILE formalism contains the spin integrals in Ref. 11 as a subset. It is therefore possible that any discrepancy between the theory in Ref. 11 and the data in Fig. 11 are due to the additional integrals contained in the SMILE formalism but not in Ref. 11.

III. CONCLUSIONS

The SMILE program has been used to calculate the polarization for a number of storage ring models, to elucidate various aspects of the behaviour of the polarization. For example, the vanishing of certain resonances due to machine symmetry, the effect of beam energy spread, the relative importance of transverse recoils, and the reduction in the polarization due to the $\langle \hat{n} \rangle$ factor (the spread in the distribution of spin quantization axes $\hat{n}(\vec{r}, \vec{p}, \theta)$ due to the spread in coordinates and momenta). Note that the program was able to calculate all combinations of orbital modes, including overlapping resonances, for machines of arbitrary geometry, including non-planar rings (Figs. 1, 2 and 8), and including closed orbit distortions. In addition, sample results were shown for a high-energy ring (PEP), and reasonable agreement was obtained for the ratios of the widths of certain spin resonances in the SPEAR data of Ref. 8 (Fig. 12).

ACKNOWLEDGEMENTS

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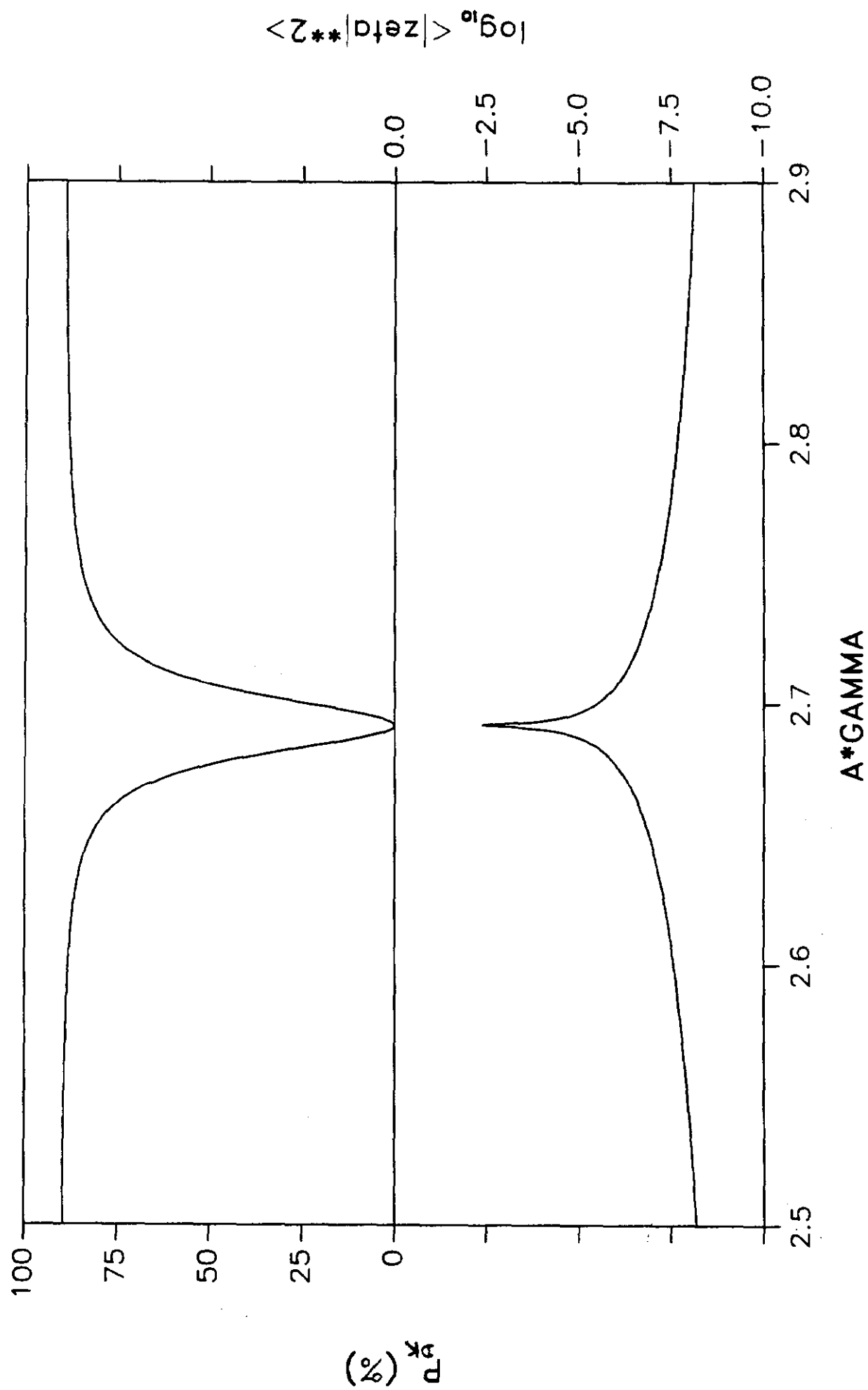12 J.R. Johnson, private communication.

Fig. 1

P(%)

A✳GAMMA

$v = 2 + Q_S$

$v = 2 + Q_z$

$v = 4 - Q_z$

Fig. 2

Fig. 3

Fig. 4

Fig. 5

P (%)

A*GAMMA ν= αz

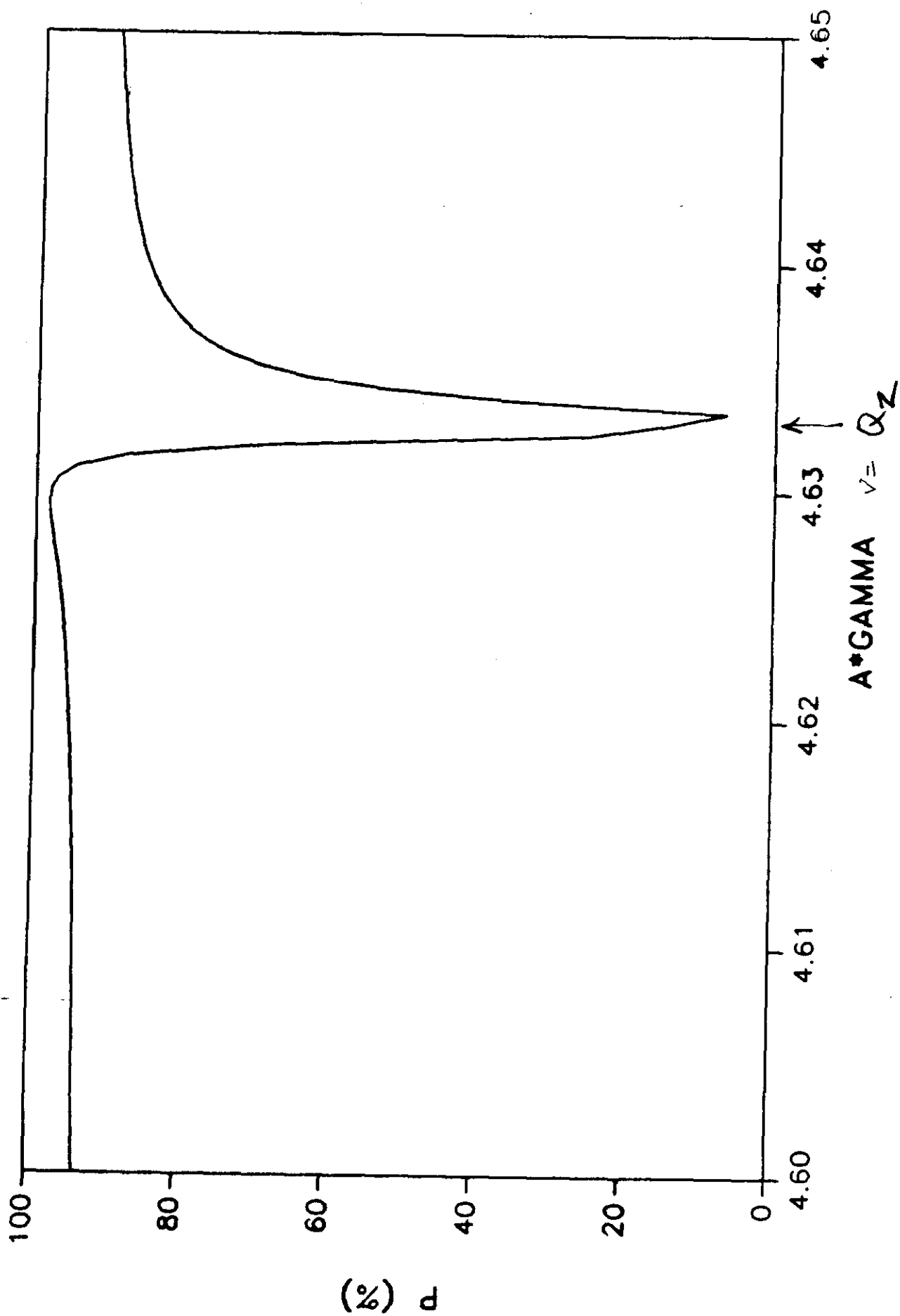Fig. 6

Fig. 7

Fig. 8

P (%) — 100, 80, 60, 40, 20, 0

A*GAMMA — 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, 3.4

$\nu=3-Q_z$

$\nu=3$

$\nu=3-Q_s$

$\nu=3+Q_s$

Fig. 19

PEP mini-beta

Fig. 10

PEP SSRL

Fig. 11

$\nu=8$ $\qquad$ $\nu-\nu_x+\nu_y=8$ $\qquad$ $\nu-\nu_y=3$ $\qquad$ $\nu-\nu_x+\nu_s=3$ $\qquad$ $\nu-\nu_x=3$ $\qquad$ $\nu-\nu_x-\nu_s=3$ $\qquad$ $\nu-\nu_x-2\nu_s=3$ $\qquad$ $\nu-\nu_x-\nu_y=-2$
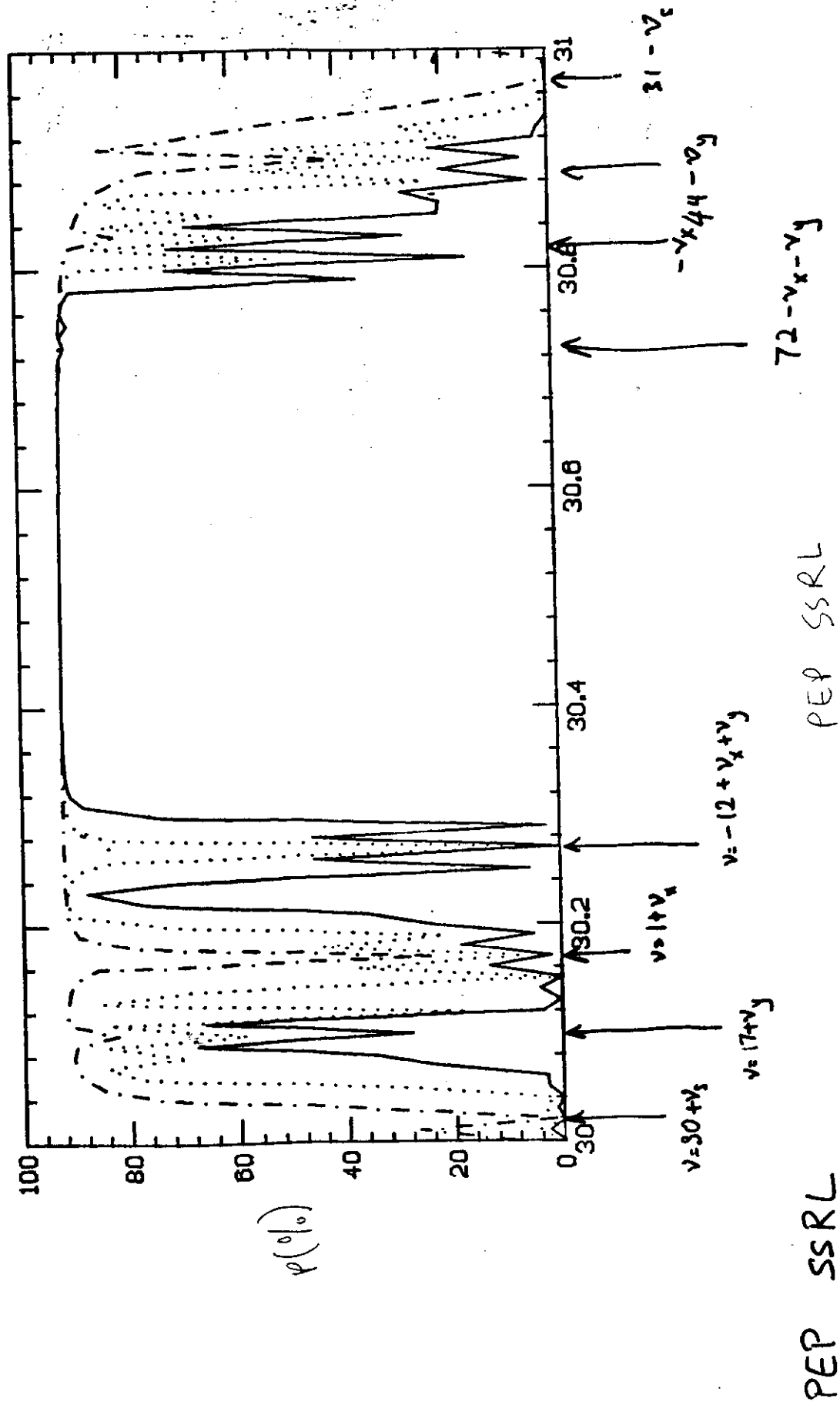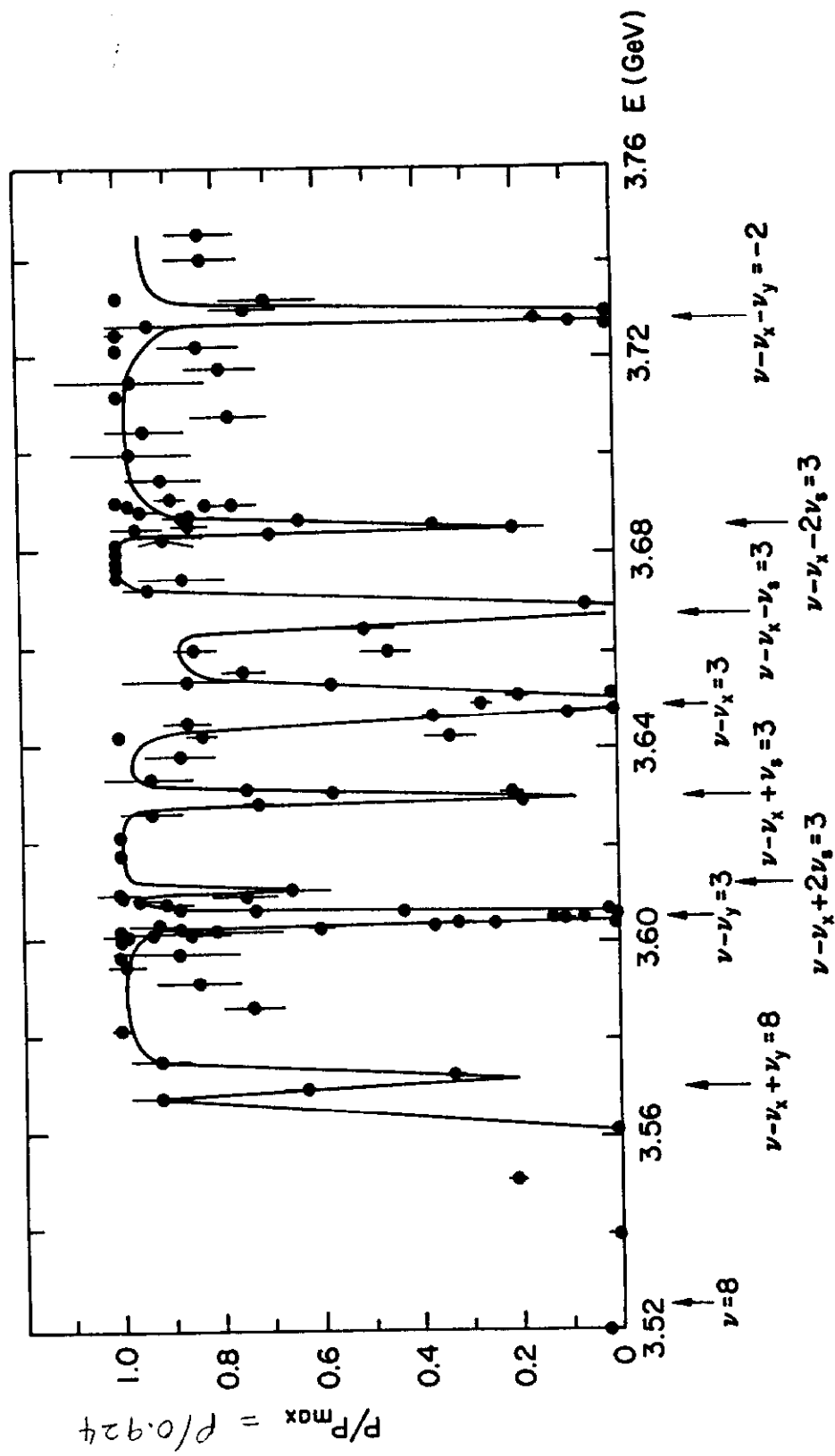
$\nu-\nu_x+2\nu_s=3$
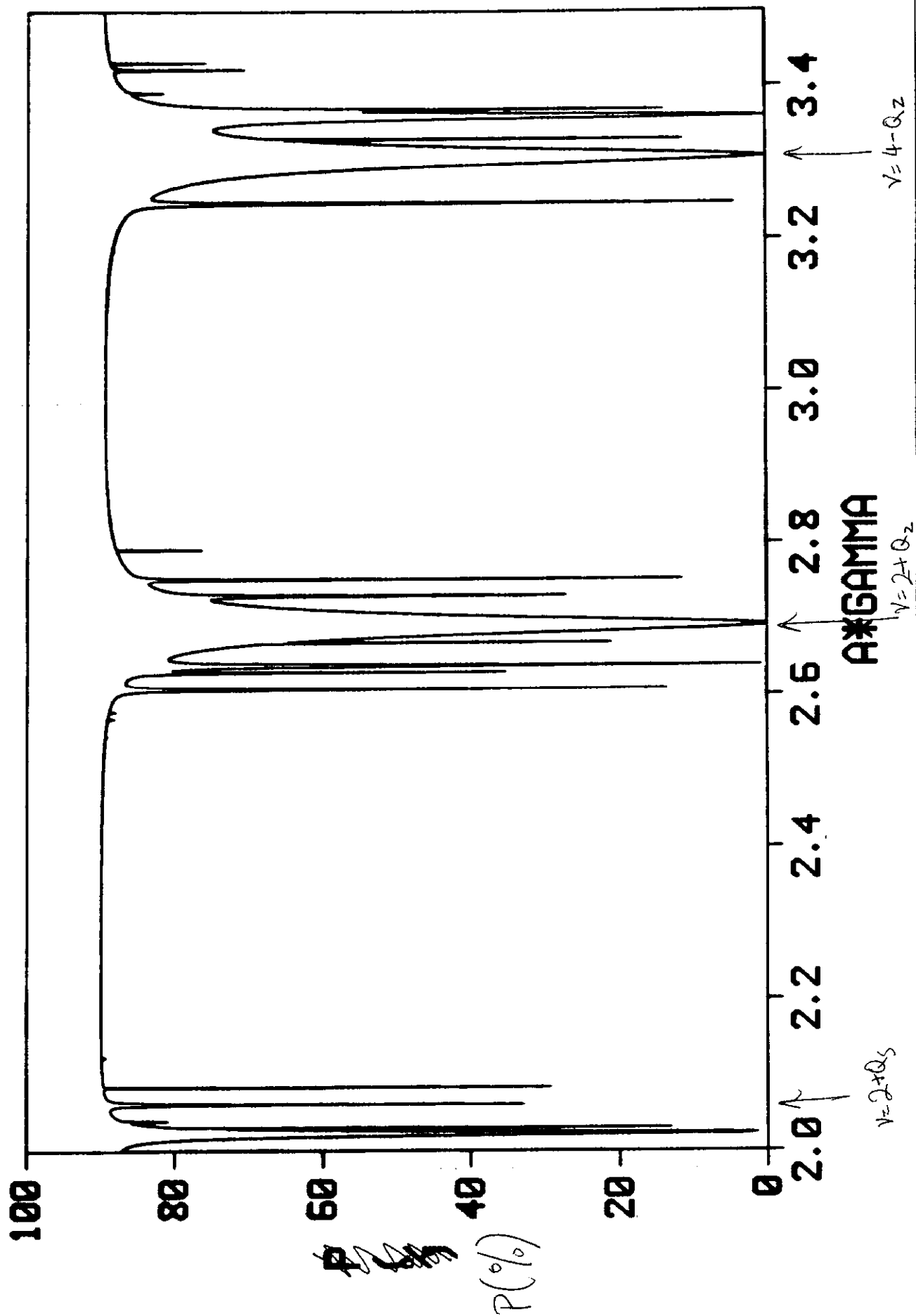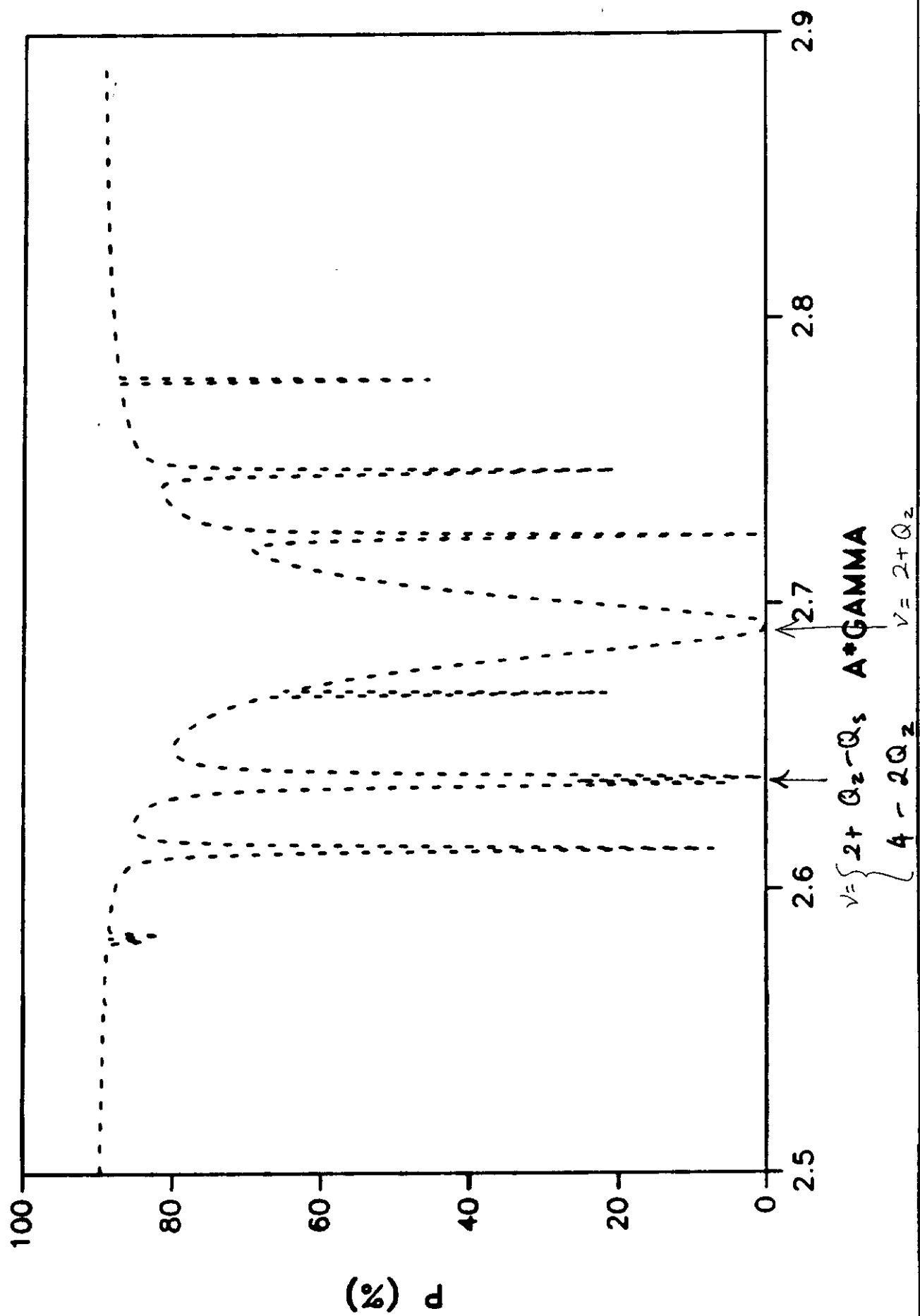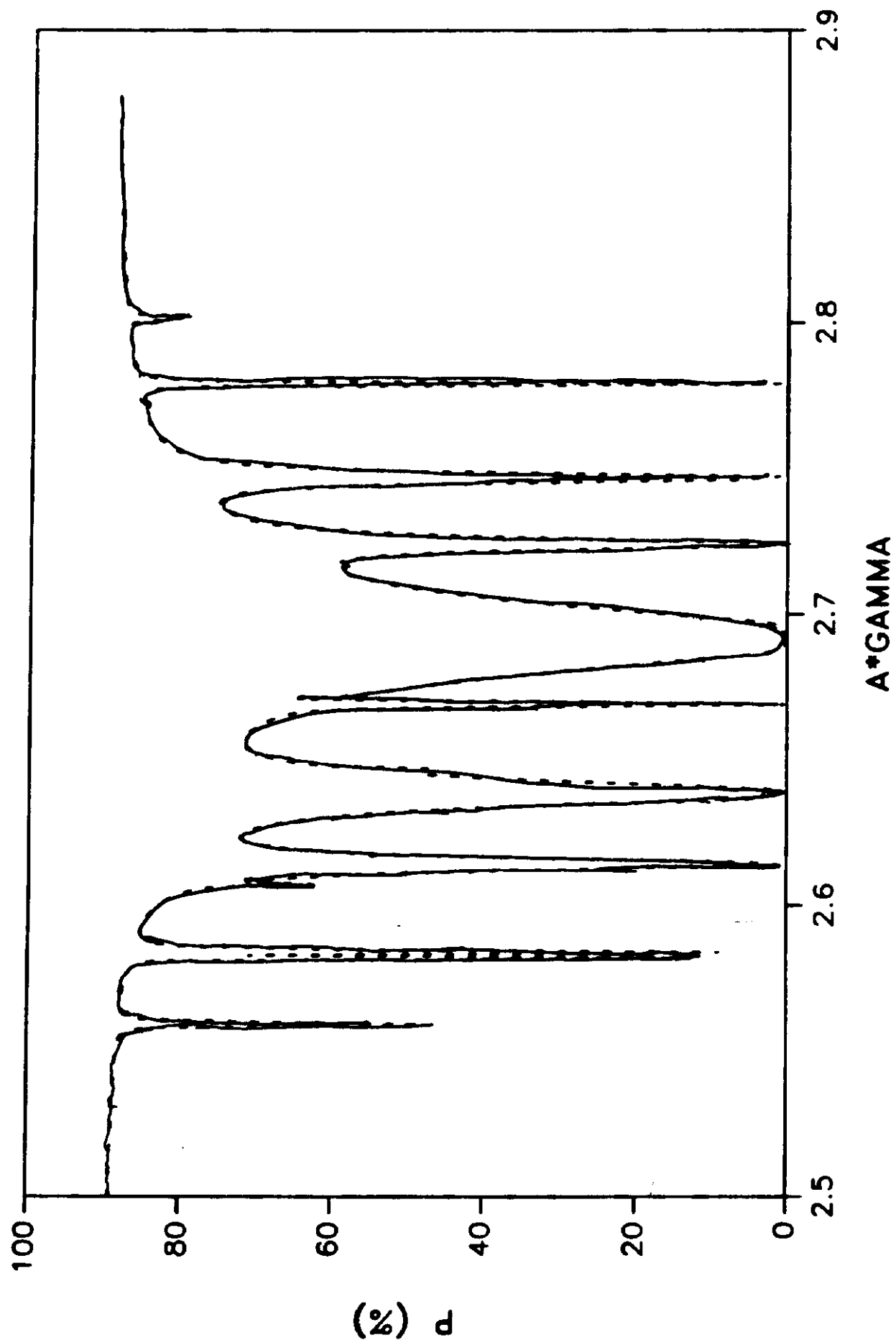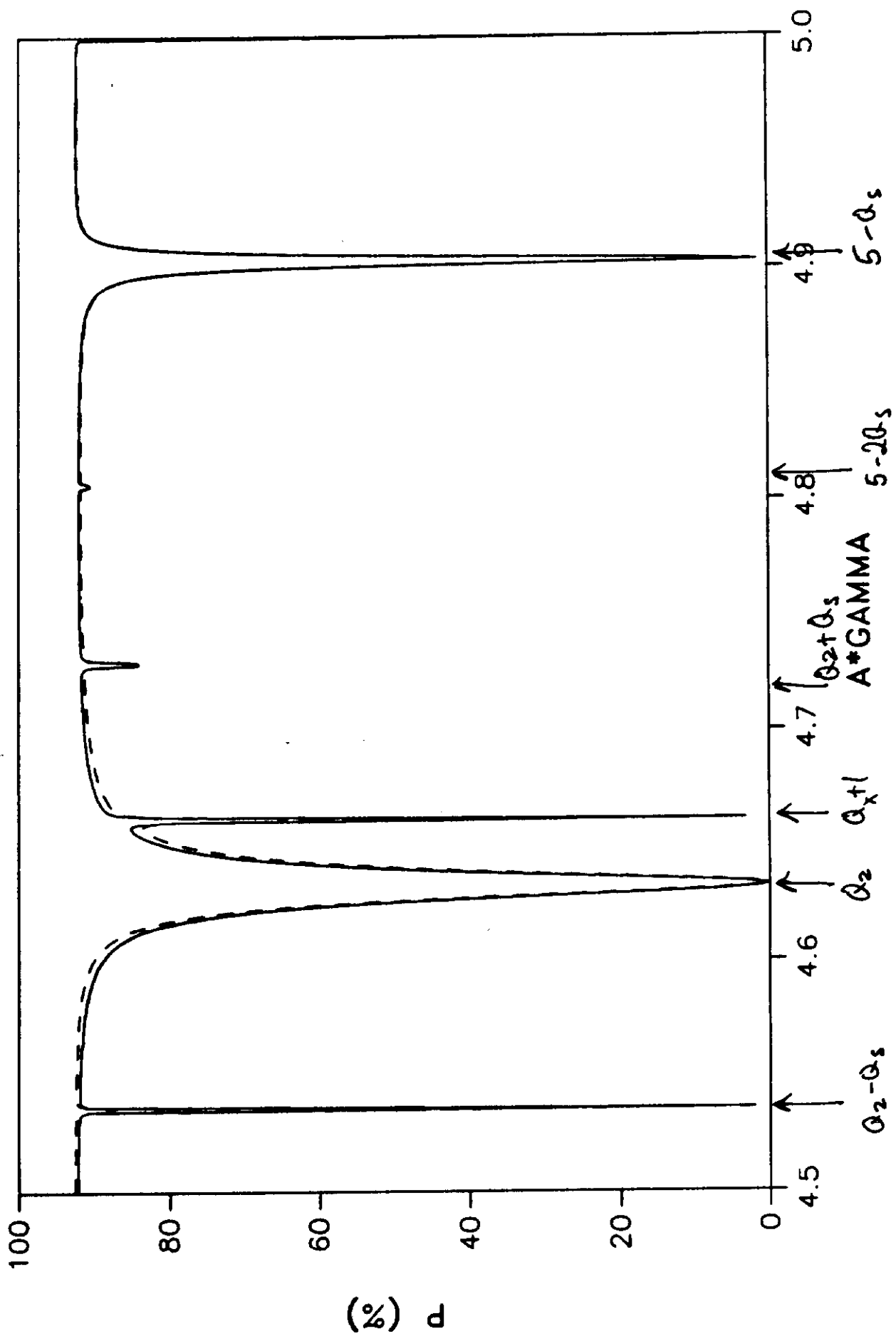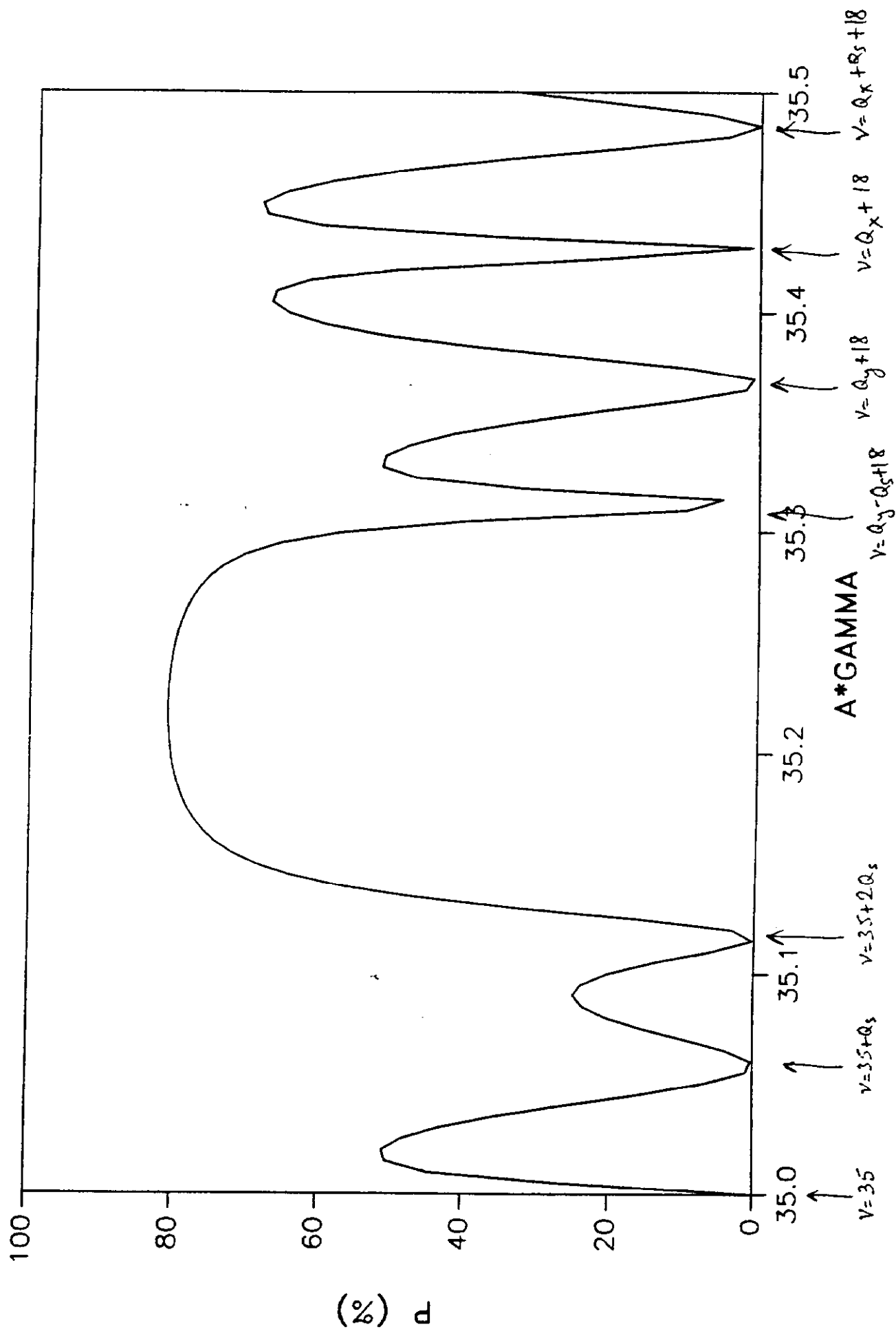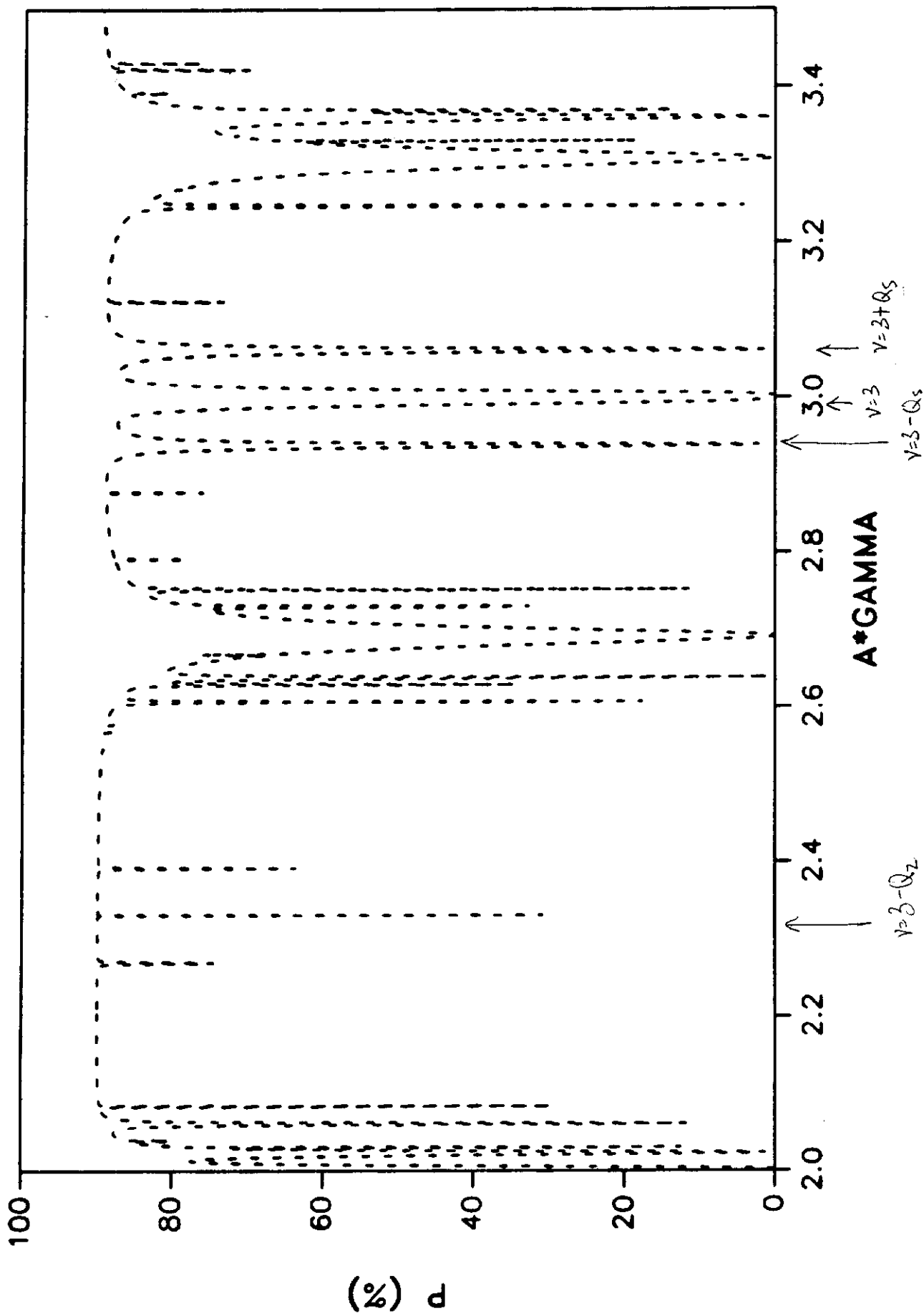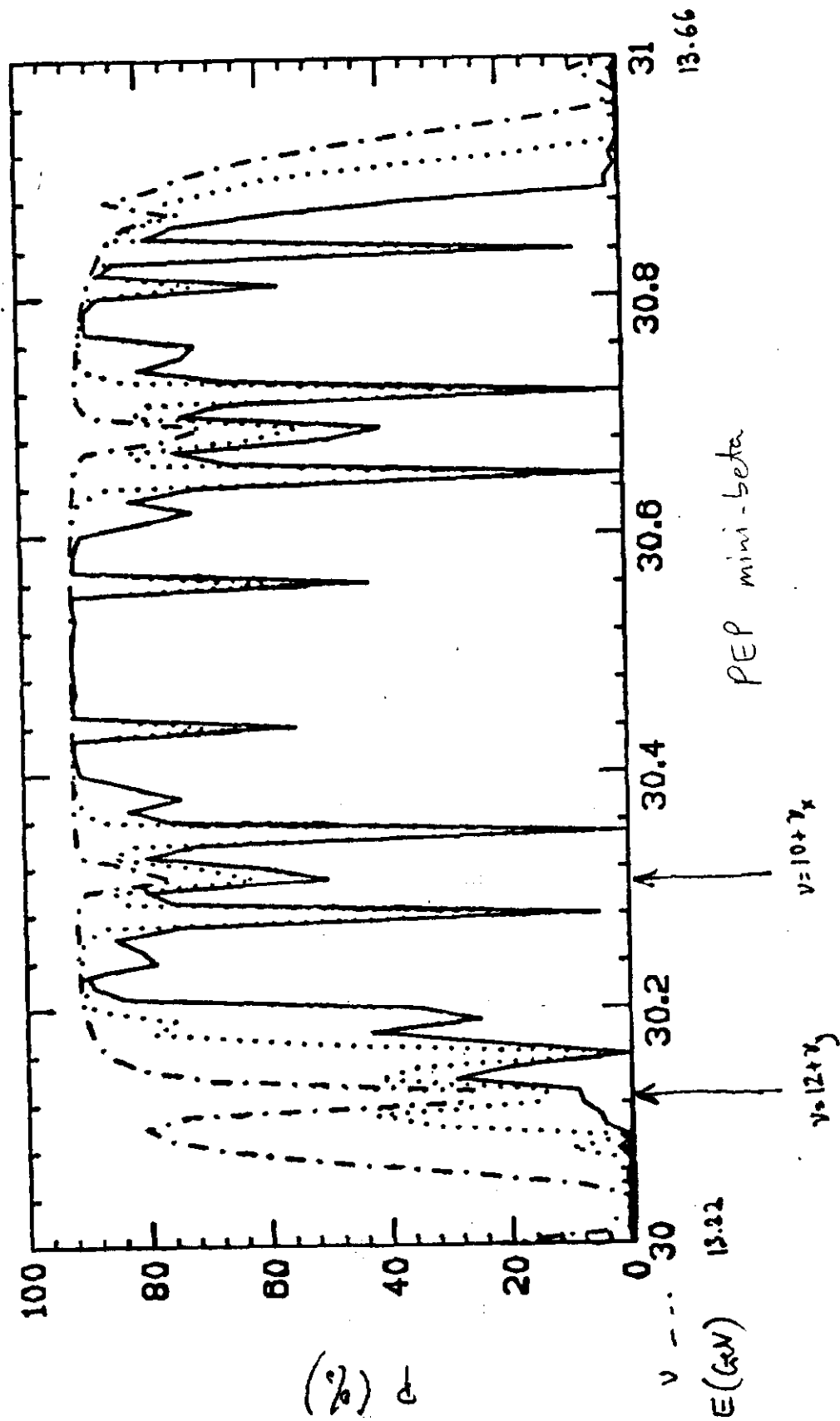
(reproduced from ref. 9)

Fig. 12